Practice USAPhO X

INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

- Work Part A first. You have 90 minutes to complete all problems. Each problem is worth an equal number of points, with a total point value of 60. Do not look at Part B during this time.
- After you have completed Part A you may take a break.
- Then work Part B. You have 90 minutes to complete all problems. Each problem is worth an equal number of points, with a total point value of 60. Do not look at Part A during this time.
- Show all your work. Partial credit will be given. Do not write on the back of any page. Do not write anything that you wish graded on the question sheets.
- Start each question on a new sheet of paper. Put your AAPT ID number, your proctor's AAPT ID number, the question number, and the page number/total pages for this problem, in the upper right hand corner of each page. For example,

  Student AAPT ID #
  Proctor AAPT ID #
  A1 – 1/3

- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones, PDA's or cameras may not be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
- Questions with the same point value are not necessarily of the same difficulty.
- In order to maintain exam security, do not communicate any information about the questions (or their answers/solutions) on this contest.

Possibly Useful Information. You may use this sheet for both parts of the exam.

$g = 9.8$ N/kg	$G = 6.67 \times 10^{-11}\ \text{N} \cdot \text{m}^2/\text{kg}^2$
$k = 1/4\pi\epsilon_0 = 8.99 \times 10^9\ \text{N} \cdot \text{m}^2/\text{C}^2$	$k_\text{m} = \mu_0/4\pi = 10^{-7}\ \text{T} \cdot \text{m/A}$
$c = 3.00 \times 10^8$ m/s	$k_\text{B} = 1.38 \times 10^{-23}$ J/K
$N_\text{A} = 6.02 \times 10^{23}\ (\text{mol})^{-1}$	$R = N_\text{A} k_\text{B} = 8.31\ \text{J}/(\text{mol} \cdot \text{K})$
$\sigma = 5.67 \times 10^{-8}\ \text{J}/(\text{s} \cdot \text{m}^2 \cdot \text{K}^4)$	$e = 1.602 \times 10^{-19}$ C
$1\ \text{eV} = 1.602 \times 10^{-19}$ J	$h = 6.63 \times 10^{-34}\ \text{J} \cdot \text{s} = 4.14 \times 10^{-15}\ \text{eV} \cdot \text{s}$
$m_e = 9.109 \times 10^{-31}\ \text{kg} = 0.511\ \text{MeV}/c^2$	$(1+x)^n \approx 1 + nx$ for $\lvert x \rvert \ll 1$
$\sin\theta \approx \theta - \frac{1}{6}\theta^3$ for $\lvert\theta\rvert \ll 1$	$\cos\theta \approx 1 - \frac{1}{2}\theta^2$ for $\lvert\theta\rvert \ll 1$

Part A

Question A1

In this problem, we analyze the working principle of a speed camera. The transmitter of the speed camera emits an electromagnetic wave of frequency $f_0 = 24\,\text{GHz}$ having waveform $\cos(2\pi f_0 t)$. The wave gets reflected from an approaching car moving with speed v. The reflected wave is recorded by the receiver of the speed camera.

1. What is the frequency f_1 of the reflected wave? You may assume that $v \ll c$.
2. Inside the circuitry of the speed camera, the received waveform is multiplied with the original emitted waveform. This product can itself be written as a sum of sinusoids with several distinct frequencies. Find all the distinct frequencies present.
3. Given that the lowest frequency component present in the multiplied signal is $f_{\text{low}} = 4.8\,\text{kHz}$, calculate the speed of the car v.

You may find it useful to use the trigonometric identity

$$\cos\alpha\cos\beta = (\cos(\alpha+\beta) + \cos(\alpha-\beta))/2.$$

Question A2

Wood burns in a fireplace on the ground, producing smoke with temperature $T_1 = 40\,^\circ\text{C}$, which slowly rises through the atmosphere. Neglect the exchange of heat between the smoke and the surrounding air, and suppose the atmosphere has pressure $p_0 = 100\,\text{kPa}$ at the ground, and uniform temperature $T_0 = 20\,^\circ\text{C}$. Treat both the smoke and atmosphere as diatomic ideal gases of molar mass $\mu = 29\,\text{g/mol}$, and recall that $0\,^\circ\text{C} = 273\,\text{K}$. To within 10%, how high does the smoke column rise?

Question A3

Consider a particle of mass m confined to a one-dimensional box of length L. We consider the quantum mechanics of this system. For simplicity, express your answers in terms of the quantity $\alpha = h^2/8m$ as much as possible.

1. In each energy level, the particle may be represented by a standing wave, where the wavefunction is zero at the walls. Find the wavelength λ for the n^{th} energy level.
2. Using the de Broglie relation $p = h/\lambda$, find the energy E_n of the n^{th} energy level.
3. Electrons are fermions, meaning that each energy level can only be occupied with two electrons (one with spin up, and one with spin down). Let there be N electrons of mass m in this box, where N is an even number. Find the lowest possible total energy U_0 of the system, i.e. the ground state energy of the system. You may neglect the Coulomb interaction between the electrons, and you may use the identity

$$\sum_{n=1}^{m} n^2 = \frac{m(m+1)(2m+1)}{6}$$

4. Write the total energy U_1 in terms of U_0 and relevant quantities when the system is in the first excited state. Do the same for the total energy U_2 of the second excited state.

5. In the ground state, find the magnitude of the force F on each wall in terms of U_0.

6. In some astrophysical systems, the size L is free to vary, and in equilibrium the outward force found in part (5) balances the inward gravitational force. We can get a rough estimate for the equilibrium value of L by equating the total energy U_0 to the total gravitational potential energy. Make a very rough estimate for the gravitational potential energy using dimensional analysis, treating the box as having uniform density. Equate this to U_0 and find a rough estimate for the equilibrium length L in terms of N and other quantities.

Part B

Question B1

A submarine of unknown nationality is traveling near the bottom of the Baltic sea, at the depth of $h = 300\,\text{m}$. Its interior is one big room of volume $V = 10\,\text{m}^3$ filled with air ($M = 29\,\text{g/mol}$) at pressure $p_0 = 100\,\text{kPa}$ and temperature $t_0 = 20\,^\circ\text{C}$. Suddenly it hits a rock and a large hole of area $A = 20\,\text{cm}^2$ is formed at the bottom of the submarine. As a result, the submarine sinks to the bottom and most of it is filled fast with water, leaving a bubble of air at increased pressure (no air escapes the submarine). The density of water is $\rho = 1000\,\text{kg/m}^3$ and free fall acceleration is $g = 9.81\,\text{m/s}^2$. Treat the air as a diatomic ideal gas.

1. What is the volume rate (in m^3/s) at which the water flows into the submarine immediately after the formation of the hole?
2. The flow rate is so large that the submarine is filled with water so fast that heat exchange between the gas and the water can be neglected, in both this part and the next part. What is the volume of the air bubble once water flow has stopped?
3. At this point, what is the change in internal energy of the air?
4. The water stream rushing into the submarine creates a turbulent flow, which ultimately causes energy to be dissipated as heat. How much total energy is dissipated to heat within the water, once the inflow has stopped due to equalized pressures?

Question B2

Consider a modification of Coulomb's law by replacing it with

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\epsilon_0}\left(\frac{1}{r^2} + \frac{\beta}{r^3}\right)\hat{\mathbf{r}}$$

where β is a constant. The usual Bohr quantization condition $L = n\hbar$ still holds. Simplify your answers as much as possible, and express them in terms of the Bohr radius $a_0 = 4\pi\epsilon_0\hbar^2/me^2$, so that $\hbar$ does not appear explicitly in your answers.

1. Find the radii r_n of the electron orbits in hydrogen, under this modified law.
2. Find the corresponding energy levels E_n.
3. Find the transition energy ΔE from $n = 2$ to $n = 1$ for this modified law. For simplicity, you may assume β is small and ignore terms of order β^2 and higher.

Question B3

A detector of radio waves is placed on the sea beach at height $h = 2\,\text{m}$ above sea level. A star, which radiates electromagnetic waves of wavelength $\lambda = 21\,\text{cm}$, begins to rise over the horizon. As a result, the detector senses alternating maxima and minima in the intensity of the waves. The waves are polarized parallel to the sea surface, which is flat.

1. Let the star be an angle α above the horizon. Determine the angles α where the detector registers intensity maxima, and minima.

2. When the star just passes the horizon (i.e. when $\alpha = 0$), is the intensity increasing or decreasing?

3. Determine the ratio of the intensity at the first maximum to the next minimum. Note that upon reflection of the wave on the water surface, the ratio of the magnitudes of the electric fields of the reflected and incident waves is

$$\frac{E_r}{E_i} = \frac{n - \cos\varphi}{n + \cos\varphi}$$

where $n = 9$ is the refractive index of water at this wavelength, and φ is the incident angle of the wave from the normal.

4. Does the ratio of intensities of consecutive maxima and minima increase or decrease as the star rises?